Lie Particle And Its Batalin-Tyutin Extension

Subir Ghosh

Physics and Applied Mathematics Unit,
Indian Statistical Institute,
203 B. T. Road, Calcutta 700108, India
E-mail address: sghosh@isical.ac.in

And

Probir Pal
Physics Department,
Uluberia College,
Uluberia, Howrah 711315, India.
E-mail address: probirkumarpal@rediffmail.com

Abstract:
In this Letter we have proposed a point particle model that generates a noncommutative three-space, with the coordinate brackets being Lie algebraic in nature, in particular isomorphic to the angular momentum algebra. The work is in the spirit of our earlier works in this connection, *i.e.* PLB 618 (2005)243 and PLB 623 (2005)251, where the κ-Minkowski form of noncomutative spacetime was considered. This non-linear and operatorial nature of the configuration space coordinate algebra can pose problems regarding its quantization. This prompts us to embed the model in the Batalin-Tyutin extended space where the equivalent model comprises of phase space variables satisfying a canonical algebra. We also compare our present model with the point particle model, previously proposed by us, in the context of κ-Minkowski spacetime.

I. *Introduction*:

In the present Letter, we continue to study general quantum spaces along the lines proposed in [1, 2]. In particular, we provide a Hamiltonian system, in the point particle approach [1, 2] that simulates a Non-Commutative (NC) space where the operatorial form of NC has a Lie algebraic structure,

$$\{x^i, x^j\} = \kappa \epsilon^{ijk} x^k \ , \quad \{x^i, t\} = 0, \tag{1}$$

where ϵ^{ijk} is the fully anti-symmetric Levi-Civita tensor, with $\epsilon^{123} = 1$. Here x^i denotes the three dimensional configuration space and κ is the NC parameter. Notice that Jacobi identity corresponding to this NC algebra is satisfied. This, in fact, is a weaker condition than the criteria of noncommutativity set by [3], where double commutators in the configuration space algebra are supposed to vanish.

Indeed, NC spaces (or more generally NC spacetimes) are here to stay (for reviews see [4]). It all started in quantum field theory when Snyder [5] introduced NC spacetime to regularize the short distance singularity in a Lorentz covariant way. The recent excitement in NC spacetime physics is generated from the seminal work of Seiberg and Witten [6] who explicitly demonstrated the emergence of NC manifold in certain low energy limit of open strings moving in the background of a two form gauge field. In the quantum gravity scenario, very general considerations in black hole physics lead to the notion of a fuzzy or Non-Commutative (NC) spacetime which can avoid the paradoxes one faces in trying to localize a spacetime point within the Planck length [4]. This is also corroborated in the modified Heisenberg uncertainty principle that is obtained in string scattering results. In these instances, the NC spacetime is expressed by the Poisson bracket algebra (to be interpreted as commutators in the quantum analogue),

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}, \tag{2}$$

where noncommutativity is of the simplest form, $\theta^{\mu\nu}$ being a *c*-number constant. From the above well known examples, it might appear that NC physics is relevant only in relativistic quantum theories and that too at the Planck regime. However, this is far from the truth as the recent shift of focus of NC physics in condensed matter systems clearly indicates. NC Chern Simons theory as an effective field theory for fractional quantum effect has been established long ago [7]. The latest excitement is in the context of an observable evidence of a magnetic monopole structure in crystal momentum space of Bloch electrons, that has appeared in the works of Onoda and Nagaosa [8] and Fang et al. [9]. These results have found an elegant interpretation in the NC phase space models Berard and Mohrbach [10]. A recent work by Duval et al. [11] have explained Berry curvature effect in Bloch electron dynamics suggested by Xiao et al. [12] through an NC phase space. The relevance of these ideas in the context of *spintronics* [13], a spin Hall effect which might induce a spin current in semiconductor samples, has been elaborated in [14].

It is very significant that the NC effects in condensed matter systems have brought in to fore their relevance in non-relativistic physics and at the same time have emphasized the importance of more general (*operatorial*) forms of noncommutativity that is present in the latter systems. This is quite in contrast to the simple constant form of NC spacetime (2) that was prevalent before.

However, the theoretical framework for NC spacetimes, with a Lie algebraic form of structure constants $C_\lambda^{\mu\nu}$

$$\{x^\mu, x^\nu\} = C_\lambda^{\mu\nu} x^\lambda \tag{3}$$

has already been developed in [15]. It is important to note that the NC extension in [3, 17] is *operatorial* and do not jeopardize the Lorentz invariance in relativistic models, which is the case with (2) with constant $\theta^{\mu\nu}$. (For an introduction to this subject the readers are referred to [16].) Of particular importance in the above is a restricted class of spacetimes known as κ-Minkowski spacetime (or κ-spacetime in short), that is described by the algebra,

$$\{x^i, t\} = kx^i \quad , \quad \{x^i, x^j\} = \{t, t\} = 0. \tag{4}$$

In the above, x^i and t denote the space and time operators respectively. Some of the important works in this topic that discusses, among other things, construction of a quantum field theory in κ-spacetime, are provided in [17]. Very interestingly, Amelino-Camelia [18] has proposed an alternative path to quantum gravity - "the doubly special relativity" - in which *two* observer independent parameters, (the velocity of light and Planck's constant), are present. It has been shown [19] that κ-spacetime is a realization of the above. Our previous works [1, 2] along these lines provide a point particle model for the above spacetime. Indeed, it has been rightly pointed out by Amelino-Camelia in [16] that of all the different forms of Lie algebraic NC structures, κ-Minkowski algebra (4) is probably the simplest. This fact will also be amply demonstrated at the end of our work, where we compare and contrast between the present particle model and the one formulated for κ-spacetime in [1, 2].

Let us put our work in the proper perspective. Since the operatorial forms NC spaces (or times) are playing increasingly dominant roles in non-relativistic (condensed matter and other) systems which are accessible to experimental verifications, it becomes imperative that one should have an intuitive (particle like) model that is equivalent to the above in some sense. In a non-relativistic setup, NC space, originating from the lowest Landau level projection of charged particles moving in a plane under the influence of a uniform, perpendicular (and strong) magnetic field [4], has become the prototype of a simple physical system (qualitatively) describing considerably more complex and abstract phenomena, in this case open strings moving in the presence of a background two form gauge field [6] mentioned before. This sort of intuitive picture, if present, is very useful and appealing. We provided this mechanism in [1, 2] for the κ-Minkowski spacetime. The present work deals with the Lie algebraic form of NC space as given in (1).

The model we propose has Hamiltonian constraints in the Dirac formalism [20] (or equivalently it has a non-trivial symplectic structure [21]). The resulting Dirac brackets [20] provide the NC space algebra in a classical setup. Quite obviously, the non-linear and operatorial form of Dirac brackets obtained here are not amenable to a conventional quantization where one elevates the status of Dirac brackets to commutators via the correspondence principle. This problem is tackled in the Batalin-Tyutin (BT) formalism [22] where one embeds the physical system in a suitably extended phase space such that one has a dual system in a completely *canonical* phase space. This removes the ambiguities in the prescription for canonical quantization in this type of models. The BT embedding is quite involved due to the non-linear nature of the constraints and we explicitly discuss the results for κ and BT auxiliary Degrees Of Freedom (DOF) in the lowest non-trivial order. Other examples of BT embedding for non-linear systems can be found in [23].

The paper is organized as follows: In Section II we put forward the non-relativistic particle model that has an underlying symplectic structure that simulates the NC coordinate space algebra, isomorphic to the angular momentum Lie algebra. Section III is devoted to the BFT embedding of our model. In Section IV we put forward a comparison between the proposed particle models (in this paper as well as in the previous ones [1, 2]) referring to different Lie algebraic structures of noncommutativity. The paper ends with a Conclusion in Section V.

II. The Lie particle:

We posit the following first order Lagrangian

$$L = \dot{\eta}^k(x^k + \frac{\kappa}{2}\epsilon^{ijk}x^i\eta^j) - V, \tag{5}$$

describing the "Lie particle" that generates the spatial angular momentum algebra we are interested in. The potential V is so far unspecified. The phase space variables are

$$p^i = 0 \ ; \quad \pi^i = x^i + \frac{\kappa}{2}\epsilon^{ijk}x^j\eta^k, \tag{6}$$

with the Poisson brackets

$$\{x^i, p^j\} = \{\eta^i, \pi^j\} = \delta^{ij} \ , \quad \{(x^i, p^i), (\eta^j, \pi^j)\} = 0. \tag{7}$$

Let us quickly review the Hamiltonian constraint analysis as formulated by Dirac [20]. In this scheme the constraints are termed as First Class Constraints (FCC) if they commute modulo constraints (in the Poisson Bracket sense) or Second Class Constraints (SCC) if they do not. The FCCs induce gauge invariance in the theory whereas the SCCs tend to modify the symplectic structure of the phase space for compatibility with the SCCs. The above modification induces a replacement of the Poisson brackets by Dirac Brackets (DB) as defined below,

$$\{A, B\}_{DB} = \{A, B\} - \{A, \psi_\alpha\}(\Psi_{\alpha\beta})^{-1}\{\psi_\beta, B\}. \tag{8}$$

where ψ_α refer to the SCCs and $\Psi_{\alpha\beta} \equiv \{\psi_\alpha, \psi_\beta\}$ is invertible on the constraint surface. The Dirac brackets are compatible with the SCCs so that the SCCs can be put "strongly" to zero.

According to the above classification [20] the set of constraints obtained from (6),

$$\psi_1^i = p^i \ , \quad \psi_2^i = \pi^i - x^i - \frac{\kappa}{2}\epsilon^{ijk}x^j\eta^k, \tag{9}$$

comprise an SCC system since they obey a non-vanishing Poisson bracket structure given by,

$$\Psi_{\alpha\beta}^{ij} = \{\psi_\alpha^i, \psi_\beta^j\} \ ; \quad \alpha, \beta \equiv 1, 2 \tag{10}$$

where

$$\Psi_{\alpha\beta}^{ij} = \begin{pmatrix} 0 & \delta^{ij} - \frac{\kappa}{2}\epsilon^{ijk}\eta^k \\ -\delta^{ij} - \frac{\kappa}{2}\epsilon^{ijk}\eta^k & \kappa\epsilon^{ijk}x^k \end{pmatrix}.$$

The inverse is computed in a straightforward way,

$$\Psi_{\alpha\beta}^{(-1)ij} = \begin{pmatrix} \frac{\kappa}{1+\frac{\kappa^2}{4}\eta^2}(\epsilon^{ijk}x^k + \frac{\kappa}{2}(\eta^i x^j - \eta^j x^i)) & -\frac{1}{1+\frac{\kappa^2}{4}\eta^2}(\delta^{ij} - \frac{\kappa}{2}\epsilon^{ijk}\eta^k + \frac{\kappa^2}{4}\eta^i\eta^j) \\ \frac{1}{1+\frac{\kappa^2}{4}\eta^2}(\delta^{ij} - \frac{\kappa}{2}\epsilon^{ijk}\eta^k + \frac{\kappa^2}{4}\eta^i\eta^j) & 0 \end{pmatrix}.$$

From the definition (8) this leads to the Dirac brackets,

$$\{x^i, x^j\}_{DB} = \frac{\kappa}{1 + \frac{\kappa^2}{4}\eta^2}(\epsilon^{ijk}x^k + \frac{\kappa}{2}(\eta^i x^j - \eta^j x^i)), \tag{11}$$

$$\{\eta^i, \pi^j\}_{DB} = \delta^{ij} \; ; \quad \{\eta^i, \eta^j\}_{DB} = 0,$$

$$\{x^i, \eta^j\}_{DB} = -\frac{1}{1 + \frac{\kappa^2}{4}\eta^2}(\delta^{ij} - \frac{\kappa}{2}\epsilon^{ijk}\eta^k + \frac{\kappa^2}{4}\eta^i\eta^j)$$

$$\{x^i, \pi^j\}_{DB} = \frac{\kappa}{2(1 + \frac{\kappa^2}{4}\eta^2)}(\epsilon^{ijk}x^k + \frac{\kappa}{2}((x.\eta)\delta^{ij} - x^i\eta^j) - \frac{\kappa^2}{4}\eta^i\epsilon^{jkl}\eta^k x^l). \tag{12}$$

In order to focus on the salient features we restrict ourselves to results valid up to the lowest non-trivial order in κ and obtain the NC algebra

$$\{x^i, x^j\}_{DB} = \kappa\epsilon^{ijk}x^k + O(\kappa^2) , \tag{13}$$

$$\{x^i, \eta^j, \}_{DB} = -\delta^{ij} + \frac{\kappa}{2}\epsilon^{ijk}\eta^k + O(\kappa^2)$$

$$\{x^i, \pi^j\}_{DB} = \frac{\kappa}{2}\epsilon^{ijk}x^k + O(\kappa^2); \; \{\eta^i, \eta^j\}_{DB} = 0 \; ; \; \{\eta^i, \pi^j\}_{DB} = \delta^{ij} + O(\kappa^2) \tag{14}$$

Notice that to $O(\kappa)$ we have recovered the angular momentum algebra for the coordinates. This is one of our main results.

Let us note an important point related to the applicability of our formalism for the general Lie algebraic structure, as given in (3). It is quite clear that the presence of ϵ^{ijk} in the κ-term of the Lagrangian in (5) is responsible for the cherished Dirac Bracket of (13):

$$\{x^i, x^j\}_{DB} = \kappa\epsilon^{ijk}x^k + O(\kappa^2).$$

In order to reproduce the general form of Lie algebra among x^i, as given in (3), one has to modify the Lagrangian accordingly. Hence, in principle, there is no problem in generalizing our model for the general Lie algebraic noncommutativity.

We fix the form of potential V to get an idea about the dynamics and a natural choice is to consider a harmonic oscillator potential,

$$H = V = x^2 + \nu\eta^2 \tag{15}$$

which is scaled by an overall dimensional parameter apart from the constant ν. The equations of motion for a generic variable O is

$$\dot{O} = \{O, H\}_{DB}. \tag{16}$$

In the present case we find

$$\dot{x}^i = -2\nu\eta^i , \quad \dot{\eta}^i = 2x^i - \kappa\epsilon^{ijk}x^j\eta^k. \tag{17}$$

Iterating the above equations one more time we get

$$\ddot{x}^i = -4\nu x^i + 2\nu\kappa\epsilon^{ijk}x^j\eta^k , \quad \ddot{\eta}^i = -4\nu\eta^i. \tag{18}$$

Note that we will get harmonic oscillator equations of motion for both x^i and η^i only if $\nu = \kappa$.

It is well-known from the theory of differential forms (Darboux's theorem) that for a symplectic manifold, by a suitable transformation locally one can always go to a set of coordinates which are canonical. In the present case associativity, that is validity of the Jacobi identity, among phase space variables is assured since we have used Dirac brackets. The Darboux coordinates are defined to be,

$$Q^i \equiv x^i + \frac{\kappa}{2}\epsilon^{ijk}x^j\eta^k \; ; \quad P^i \equiv -\eta^i. \tag{19}$$

They obey the canonical algebra

$$\{Q^i, Q^j\} = \{P^i, P^j\} = 0 \; ; \quad \{Q^i, P^j\} = \delta^{ij}. \tag{20}$$

III. *BT embedding of the Lie particle*:

We start by a brief digression on the BT formalism [22]. The basic idea is to embed the original system in an enlarged phase space (the BT space), consisting of the original "physical" degrees of freedom and auxiliary variables, in a particular way such that the resulting enlarged system possesses local gauge invariance. Imposition of gauge conditions accounts for the true number of degrees of freedom and at the same time the freedom of having different gauge choices leads to structurally distinct systems. However, all of them are assured to be gauge equivalent.

Let us consider a generic set of constraints (ψ_α, ξ_l) and a Hamiltonian operator H with the following PB relations,

$$\{\psi_\alpha(q), \psi_\beta(q)\} \approx \Delta_{\alpha\beta}(q) \neq 0 \; ; \quad \{\psi_\alpha(q), \xi_l(q)\} \approx 0$$

$$\{\xi_l(q), \xi_n(q)\} \approx 0 \; ; \quad \{\xi_l(q), H(q)\} \approx 0. \tag{21}$$

In the above (q) collectively refers to the set of variables present prior to the BT extension and "$\approx$" means that the equality holds on the constraint surface. Clearly ψ_α and ξ_l are SCC and FCC respectively.

In systems with non-linear SCCs, (such as the present one), in general the DBs can become dynamical variable dependent due to the $\{A, \psi_\alpha\}$ and $\Delta_{\alpha\beta}$ terms, leading to problems for the quantization programme. This type of pathology is cured in the BT formalism in a systematic way, where one introduces the BT variables ϕ^α, obeying

$$\{\phi^\alpha, \phi^\beta\} = \omega^{\alpha\beta} = -\omega^{\beta\alpha}, \tag{22}$$

where $\omega^{\alpha\beta}$ is a constant (or at most a c-number function) matrix, with the aim of modifying the SCC $\psi_\alpha(q)$ to $\tilde{\psi}_\alpha(q, \phi^\alpha)$ such that,

$$\{\tilde{\psi}_\alpha(q, \phi), \tilde{\psi}_\beta(q, \phi)\} = 0 \; ; \quad \tilde{\psi}_\alpha(q, \phi) = \psi_\alpha(q) + \Sigma_{n=1}^\infty \tilde{\psi}_\alpha^{(n)}(q, \phi) \; ; \quad \tilde{\psi}^{(n)} \approx O(\phi^n) \tag{23}$$

This means that $\tilde{\psi}_\alpha$ are now FCCs and in particular abelian. The explicit terms in the above expansion are,

$$\tilde{\psi}_\alpha^{(1)} = X_{\alpha\beta}\phi^\beta \; ; \quad \Delta_{\alpha\beta} + X_{\alpha\gamma}\omega^{\gamma\delta}X_{\beta\delta} = 0 \tag{24}$$

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$$\tilde{\psi}_{\alpha}^{(n+1)} = -\frac{1}{n+2}\phi^{\delta}\omega_{\delta\gamma}X^{\gamma\beta}B_{\beta\alpha}^{(n)} \quad ; \quad n \geq 1 \tag{25}$$

$$B_{\beta\alpha}^{(1)} = \{\tilde{\psi}_{\beta}^{(0)}, \tilde{\psi}_{\alpha}^{(1)}\}_{(q)} - \{\tilde{\psi}_{\alpha}^{(0)}, \tilde{\psi}_{\beta}^{u(1)}\}_{(q)} \tag{26}$$

$$B_{\beta\alpha}^{(n)} = \Sigma_{m=0}^{n}\{\tilde{\psi}_{\beta}^{(n-m)}, \tilde{\psi}_{\alpha}^{(m)}\}_{(q,p)} + \Sigma_{m=0}^{n}\{\tilde{\psi}_{\beta}^{(n-m)}, \tilde{\psi}_{\alpha}^{(m+2)}\}_{(\phi)} \quad ; \quad n \geq 2 \tag{27}$$

In the above, we have defined,

$$X_{\alpha\beta}X^{\beta\gamma} = \omega_{\alpha\beta}\omega^{\beta\gamma} = \delta_{\alpha}^{\gamma}. \tag{28}$$

A very useful idea is to introduce the improved function $\tilde{f}(q)$ corresponding to each $f(q)$,

$$\tilde{f}(q, \phi) \equiv f(\tilde{q}) = f(q) + \Sigma_{n=1}^{\infty}\tilde{f}(q, \phi)^{(n)} \quad ; \quad \tilde{f}^{(1)} = -\phi^{\beta}\omega_{\beta\gamma}X^{\gamma\delta}\{\psi_{\delta}, f\}_{(q)} \tag{29}$$

$$\tilde{f}^{(n+1)} = -\frac{1}{n+1}\phi^{\beta}\omega_{\beta\gamma}X^{\gamma\delta}G(f)_{\delta}^{\lambda(n)} \quad ; \quad n \geq 1 \tag{30}$$

$$G(f)_{\beta}^{(n)} = \Sigma_{m=0}^{n}\{\tilde{\psi}_{\beta}^{(n-m)}, \tilde{f}^{(m)}\}_{(q)} + \Sigma_{m=0}^{(n-2)}\{\tilde{\psi}_{\beta}^{(n-m)}, \tilde{f}^{(m+2)}\}_{(\phi)} + \{\tilde{\psi}_{\beta}^{(n+1)}, \tilde{f}^{(1)}\}_{(\phi)} \tag{31}$$

which have the property $\{\tilde{\psi}_{\alpha}(q, \phi), \tilde{f}(q, \phi)\} = 0$. Thus, in the BT space, the improved functions are FC or equivalently gauge invariant. Note that $\tilde{q}$ corresponds to the improved variables for q. The subscript (ϕ) and (q) in the PBs indicate the variables with respect to which the PBs are to be taken. It can be proved that extensions of the original FCC ξ_l and Hamiltonian H are simply,

$$\tilde{\xi}_l = \xi(\tilde{q}) \quad ; \quad \tilde{H} = H(\tilde{q}). \tag{32}$$

One can also reexpress the converted SCCs as $\tilde{\psi}_{\alpha}^{\mu} \equiv \psi_{\alpha}^{\mu}(\tilde{q})$. The following identification theorem holds,

$$\{\tilde{A}, \tilde{B}\} = \{A, \tilde{B}\}_{DB} \quad ; \quad \{\tilde{A}, \tilde{B}\}\mid_{\phi=0} = \{A, B\}_{DB} \quad ; \quad \tilde{0} = 0. \tag{33}$$

Hence the outcome of the BT extension is the closed system of FCCs with the FC Hamiltonian given below,

$$\{\tilde{\psi}_{\alpha}, \tilde{\psi}_{\beta}\} = \{\tilde{\psi}_{\alpha}, \tilde{\xi}_l\} = \{\tilde{\psi}_{\alpha}^{\mu}, \tilde{H}\} = 0 \quad ; \quad \{\tilde{\xi}_l, \tilde{\xi}_n\} \approx 0 \; ; \; \{\tilde{\xi}_l, \tilde{H}\} \approx 0. \tag{34}$$

In general, due to the non-linearity in the SCCs, the extensions in the improved variables, (and subsequently in the FCCs and FC Hamiltonian), may turn out to be infinite series. This type of situation has been encountered before [23].

Let us concentrate on the problem at hand. The BT variables satisfy the canonical algebra

$$\{\phi_{i\alpha}, \phi_{j\beta}\} = \delta_{ij}\epsilon_{\alpha\beta} \equiv \omega_{ij\alpha\beta} \quad ; \quad \epsilon^{01} = -\epsilon_{01} = 1. \tag{35}$$

The structure of $\tilde{O}$ for a generic variable O is of the form

$$\tilde{O} = O + O^{(1)}(\phi) + O(\phi\phi, ...). \tag{36}$$

Truncating the above up to a single BT variable contribution we compute the FC counterparts of all the degrees of freedom:

$$\tilde{x}^i = x^i - \phi^{1i} + \frac{\kappa}{4}\epsilon^{ijk}\phi^{1j}\eta^k + \frac{\kappa}{2}\epsilon^{ijk}\phi^{2j}x^k$$

$$\tilde{\eta}^i = \eta^i + \phi^{2i} + \frac{\kappa}{4}\epsilon^{ijk}\phi^{2j}\eta^k$$

$$\tilde{p}^i = p^i + \phi^{2i} - \frac{\kappa}{4}\epsilon^{ijk}\phi^{2j}\eta^k \;, \;\; \tilde{\pi}^i = \pi^i + \frac{\kappa}{2}\epsilon^{ijk}\phi^{2j}x^k \tag{37}$$

The corresponding FC Hamiltonian will be

$$\tilde{H} = \tilde{x}^2 + \nu\tilde{\eta}^2. \tag{38}$$

This Hamiltonian together with the canonical set of phase space variables in the BT extended space can be quantized in the conventional way.

IV. *Comparison between Lie particle models*:

In this section, we will now compare and contrast some of the features of the point particle model in [1] for the κ-spacetime and the particle model formulated here. In both cases, we started with an extended phase space having a single scalar variable η in [1] and a vector η^i in the present model, as auxiliary degrees of freedom. However, in [1], in the reduced space, one can incorporate the constraints strongly and completely eliminate η with the resulting system comprising of only (X^i, P^j)-variables. In the present model, there is no simple way of removing η^i due to the more involved constraint structure, as derived in (9).

Secondly, in the κ-Minkowski algebra (4), a possible set of Darboux transformation can be,

$$x^i \equiv X^i \;, \;\; t \equiv \kappa(X^i P^i) \;, \;\; p^i \equiv P^i, \tag{39}$$

with X^i and P^j obeying canonical brackets. One can check that all the Jacobi identities are preserved [1]. This shows that X^i and P^j constitute a consistent set of degrees of freedom. Incidentally, the angular momentum operator also remains unchanged.

But these features are not preserved for the Lie algebra (1). In fact Jacobi identities in the (X^i, P^j)-phase space get violated. This reestablishes the fact that the extension by η^i is necessary and one can not simply remove η^i and get a consistent set of variables with a symplectic algebra. For the same reason, the Darboux coordinates contain η^i in (19) in an entangled way. Also, the angular momentum operator can be formulated in terms of the Darboux variables but it will appear quite complicated in terms of original variables. The other option is to exploit the Batalin-Tyutin framework, (that we have studied in Section III), where the extended phase space is canonical and one can construct angular momentum operator (and other relevant operators) in a straightforward manner.

The above discussion reconfirms the fact that κ-Minkowski spacetime is one of the simplest forms of Lie algebraic NC spacetime and this property is inherited by the corresponding particle

[1]The only non-trivial one being $J(x^i, p^j, t) = \{\{x^i, t\}, p^j\} + \{\{t, p^j\}, x^i\} + \{\{p^j, x^i\}, t\} = 0$.

model in [1]. On the other hand, some of these pleasant features are lost in the present point particle model that generates the configuration space algebra (1).

V. *Conclusion*:

First of all, let us summarize our work. We have provided a non-relativistic point particle model that induces a noncommutative three-space endowed with Lie algebraic form of commutator brackets. To be specific, the coordinates obey an angular momentum algebra. This type of configuration space that has a non-trivial algebra between the coordinate variables is not suitable for a canonical quantization. To circumvent this problem we have considered instead an extension of the model in Batalin-Tyutin phase space where a dual model to our original one can be considered in a canonical phase space. We have also shown how the auxiliary variables play an essential role here in comparison to other particle model in [1] that induce κ-Minkowski spacetime.

In recent years, lot of attention is being paid to spaces (or spacetimes) having a general (operatorial) form of noncommutativity. Indeed, interesting and exciting systems in the domain of low energy condensed matter bear interpretations in terms of effective models that has underlying noncommutative space structures. We believe that our point particle approach will be able to provide some intuitive understanding of these models in a simpler setting.

References

[1] S.Ghosh and Probir Pal, Phys.Lett.B 618 (2005)243, (hep-th/0502192).

[2] S.Ghosh, Phys.Lett.B 623 (2005)251, (hep-th/0506084).

[3] S.Doplicher, K.Fredenhagen and J.E.Roberts, Phys.Lett. B331 39(1994).

[4] For reviews see for example M.R.Douglas and N.A.Nekrasov, Rev.Mod.Phys. 73(2001)977; R.J.Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162.

[5] H.S.Snyder, Phys.Rev. 71 38(1947).

[6] N.Seiberg and E.Witten, JHEP 9909(1999)032.

[7] L.Susskind hep-th/0101029; S.Hellerman M.V.Raamsdonk, hep-th/0103179.

[8] M.Onoda and N.Nagaosa, J.Phys.Soc.Jpn. 71 19(2002).

[9] Z.Fang et al., Science 302 92(2003).

[10] A.Berard and H.Mohrbach. Phys.Rev. D69 95 127701(2004)

[11] C.Duval et al., cond-mat/0506051.

[12] D.Xiao, J.Shi and Q.Niu, Phys.Rev.Lett. 95 137204(2005).

[13] J.E.Hirsch, Phys.Rev.Lett. 83 1834(1999); S.Zhang, Phys.Rev.Lett. 85 393(2000); S.Datta and B.Das, Appl.Phys.Lett. 56 665(1990).

[14] R.Shindou and K.-I Imura, cond-mat/0411105.

[15] S.Majid and H.Ruegg, Phys.Lett.B 334 (1994)348.

[16] J.Madore, S.Schraml, P.Schupp and J.Wess, Eur.Phys.J. C16 (2000) 161; G.Amelino-Camelia, *Field Theories on Canonical and Lie-Algebra Noncommutative Spacetimes*, hep-th/0205047.

[17] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Annals Phys. 243 (1995)90; P. Kosiński, J. Lukierski, P. Maślanka, Phys.Rev. D62 (2000) 025004; G.Amelino-Camelia and M.Arzano, Phys.Rev. D65 (2002) 084044; A.Agostini, G.Amelino-Camelia and F.D'Andrea, Int.J.Mod.Phys. A19 (2004) 5187-5220; M.Dimitrijevic, L.Jonke, L.Moeller, E.Tsouchnika, J.Wess and M.Wohlgenannt, Czech.J.Phys. 54 (2004) 1243; Eur.Phys.J. C31 (2003) 129.

[18] G.Amelino-Camelia, Nature 418 34(2002), gr-qc/0207049.

[19] J.Kowalski-Glikman and S.Nowak, Int.J.Mod.Phys. D12 299(2003).

[20] P.A.M.Dirac, *Lectures on Quantum Mechanics*, Yeshiva University Press, New York, 1964.

[21] L.Faddeev and R.Jackiw, Phys.Rev.Lett. 60 1692(1988).

[22] I.A.Batalin and I.V.Tyutin, Int.J.Mod.Phys. A6 3255(1992).

[23] See for example N.Banerjee, R.Banerjee and S.Ghosh, Nucl.Phys. B427 257(1994); M.-U.Park and Y.-J.Park, Int.J.Mod.Phys. A13 2179(1998); S.Ghosh, Phys. Rev. D66: 045031(2002).